Applications

- Digital signal processing
    - filtering
    - moving averages
    - auralization

- Communication Systems
    - Error detection and correction in encoded messages using check matrices & linear codes

Topics (ALA 9.5 and LAA 4.2)

- Kernel and Image (aka null space and column space of A)
- The Superpostion principle
- The matrix transpose A^T
- Adjoint systems, Cokernel and Coimage (aka $A^T\underline{y}=\underline{f}$, null and col'n space of A^T)
  aka left null space and row space of A
- The Fundamental Theorem of Linear Algebra

Null Spaces, Column Spaces, and Linear Transformations

In applications of linear algebra, subspaces of $\mathbb{R}^n$ (and general vector spaces V) typically arise from either (a) the set of all solutions to a system of linear equations of the form $A\underline{x}=0$, called a homogeneous linear system, or (b) as the span of certain specified vectors.

(a) is known as the null space description of a subspace
(b) is known as the column space or image space description of a subspace.

We will see that these are intimately related to systems of linear equations.

The Null Space of a Matrix

Consider the following system of homogeneous equations:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned} \qquad (1)$$

or in matrix form $A\underline{x}=\underline{0}$ w/ $A=\begin{bmatrix} 1 & -3 & 2 \\ -5 & 9 & 1 \end{bmatrix}$.

Recall that the set of $\underline{x}$ satisfying $A\underline{x}=\underline{0}$ is the solution set of (1). Our goal here is to relate this solution set to the matrix A (this will allow us to give a geometric interpretation to the solution of the algebraic system)

We call the set of $\underline{x}$ satisfying $A\underline{x}=\underline{0}$ the null space of A. In set notation, this is written as

$$\mathrm{Null}(A) = \{\underbrace{\underline{x}}_{\text{the set of } \underline{x}} \underbrace{:}_{\text{such that}} \underbrace{\underline{x}\in\mathbb{R}^n \text{ and } A\underline{x}=0}_{\text{these conditions are satisfied.}}\}.$$

If we think of the function $f(\underline{x})=A\underline{x}$ that maps $\underline{x}\mapsto A\underline{x}$, then $\mathrm{Null}(A)$ is the subset of $\mathbb{R}^n$ that $f(\underline{x})$ maps to $\underline{0}$.

NOTE: the null space of A is also called the kernel. We will use $\mathrm{Null}(A)$ in the class notes even though ALA uses kernel, because it is more descriptive of what $\{\underline{x}: A\underline{x}=0\}$ actually is.

Example: Is $\underline{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ in Null(A) for A described in (1)?

This is simple to test, simply evaluate $A\underline{u} = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $A\underline{u} = \underline{0}$, $\underline{u} \in \text{Null}(A)$.

Now, you may be wondering: why are we calling Null(A) the null space? That's because Null(A) is a subspace!

Let's test this out: suppose that $\underline{u}, \underline{v} \in \text{Null}(A)$, and $c, d \in \mathbb{R}$ are scalars. We need to check if $c\underline{u} + d\underline{v} \in \text{Null}(A)$ too, i.e., is it true that $A(c\underline{u} + d\underline{v}) = \underline{0}$?

Well: $$\begin{aligned} A(c\underline{u} + d\underline{v}) &= c(A\underline{u}) + d(A\underline{v}) && (\text{linearity of mat-vec mult.}) \\ &= c \cdot 0 + d \cdot 0 && (\underline{u}, \underline{v} \in \text{Null}(A) \text{ so } A\underline{u} = 0, A\underline{v} = 0) \\ &= 0. \end{aligned}$$

Yes! Null(A) is a vector space! If $A \in \mathbb{R}^{m \times n}$, then Null(A) is a subspace of $\mathbb{R}^n$ (where $\underline{x}$ lives).

This property leads to the following incredibly important superposition principle for solutions to homogeneous linear systems:

Theorem: If $\underline{u}_1, \ldots, \underline{u}_k$ are each solutions to $A\underline{u} = 0$, then so is EVERY linear combination $c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k$.

WARNING: The set of solutions to $A\underline{x} = \underline{b}$, $\underline{b} \neq \underline{0}$, is NOT a subspace!

Superposition is why we like linear systems of equations: I only need to find a few specific solutions in order to construct every possible solution via linear combinations. This has tremendously important practical consequences that we'll explore throughout the rest of the semester.

FACT: Although we are focussing on linear systems of equations of the form $A\underline{x} = \underline{b}$ here, the same ideas apply to more general linear systems, e.g., those defined on infinite dimensional vector spaces like solutions to linear differential equations, which we will see later in the course.

Describing the Null Space

There is no obvious relationship between the entries of A and $\text{Null}(A)$. Rather it is defined implicitly via the condition that $\underline{x} \in \text{Null}(A)$ if and only if $A\underline{x} = \underline{0}$. However, if we compute the general solution to $A\underline{x} = \underline{0}$, this will give us an explicit description of $\text{Null}(A)$.

This can be accomplished via Gaussian Elimination.

__Example__ Let us find a basis for $\text{Null}(A)$, for $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

We reduce $[A \mid \underline{0}]$ to row echelon form in order to write the basic variables in terms of the free variables.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_3 &= -2x_4 + 2x_5 \end{aligned}$ with x_2, x_4, x_5 free.

Next, we decompose the vector giving the general solution into a linear combination of vectors where the weights are the free variables:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{u}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{u}_3}$$

$$= x_2 \underline{u}_1 + x_4 \underline{u}_2 + x_5 \underline{u}_3 .$$

Every linear combination of $\underline{u}_1, \underline{u}_2, \underline{u}_3$ is in $\text{Null}(A)$, and $\underline{u}_1, \underline{u}_2, \underline{u}_3$ are linearly independent (why?); hence $\underline{u}_1, \underline{u}_2, \underline{u}_3$ form a basis for $\text{Null}(A)$.

We conclude that $\text{Null}(A) \subset \mathbb{R}^5$ is a subspace of dimension 3.

The Column Space of A

We have seen that we can write the matrix vector $A\underline{x}$ as the linear combination

$$A\underline{x} = x_1 \underline{a_1} + x_2 \underline{a_2} + \cdots + x_n \underline{a_n},$$

of the columns $\underline{a_1}, \ldots, \underline{a_n}$ of $A = [\underline{a_1}\ \underline{a_2}\ \cdots\ \underline{a_n}]$ weighted by the elements x_i of $\underline{x}$.

By letting the coefficients $x_1, \ldots, x_n$ vary, we can describe the the subspace spanned by the columns of A, aptly named the column space of A:

$$\begin{aligned} \mathrm{Col}(A) &= \{ \underline{b} \in \mathbb{R}^m : \underline{b} = A\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n \} \\ &= \mathrm{span}\{ \underline{a_1}, \underline{a_2}, \ldots, \underline{a_n} \} \end{aligned}$$

The Col(A) is also sometimes called the image or range space of A. Because Col(A) is defined by the span of some vectors it is immediate that it is a subspace. Note however here, $\mathrm{Col}(A) \subset \mathbb{R}^m$ (where the RHS $\underline{b}$ vectors live), not $\mathbb{R}^n$ (where Null(A) and $\underline{x}$ live).

FACT: It is immediate that $A\underline{x} = \underline{b}$ has at least one solution if and only if $\underline{b} \in \mathrm{Col}(A)$.

Example: Find a matrix A so that the set

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

is equal to Col(A). To do so, we first write W as a set of linear combinations:

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Now we set these vectors as the columns of A: $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

It then follows $\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} = W.$

The Complete Solution to $A\underline{x} = \underline{b}$

With an understanding of Null(A) and Col(A), we can completely characterize the solution set to $A\underline{x}=\underline{b}$.

Theorem: The linear system $A\underline{x} = \underline{b}$ has at least one solution $\underline{x}^*$ if and only if $\underline{b} \in \text{Col}(A)$. If this occurs, then $\underline{x}$ is a solution to $A\underline{x}=\underline{b}$ if and only if

$$\underline{x} = \underline{x}^* + \underline{n}$$

where $\underline{n} \in \text{Null}(A)$ is an element of the null space of A.

Proof: We already showed the first part of the theorem. Now suppose both $\underline{x}$ and $\underline{x}^*$ are solutions so that $A\underline{x} = \underline{b} = A\underline{x}^*$. Then their difference $\underline{n} = \underline{x} - \underline{x}^*$ satisfies

$$A\underline{n} = A(\underline{x} - \underline{x}^*) = A\underline{x} - A\underline{x}^* = \underline{b} - \underline{b} = \underline{0}$$

so that $\underline{n} \in \text{Null}(A)$. This means that $\underline{x} = \underline{x}^* + (\underline{x} - \underline{x}^*) = \underline{x}^* + \underline{n}$.

This theorem tells us that to construct the most general solution to $A\underline{x} = \underline{b}$, we only need to know a particular solution $\underline{x}^*$ and the general solution to $A\underline{n} = \underline{0}$.

This might remind you of how you solved inhomogeneous linear ordinary differential equations; again, not a coincidence! We'll see later in the semester that linear algebraic systems and linear ordinary differential equations are both examples of general linear systems.

Computing the general solution to $A\underline{x} = \underline{b}$ requires applying Gaussian Elimination first to $[A|\underline{b}]$ to get a particular solution, and then to $[A|\underline{0}]$ to characterize the null space. We have worked examples for you in the online notes.

Theorem (Summary so far): If $A \in \mathbb{R}^{m\times n}$, then the following conditions are equivalent (any one implies all of the others):

(i) $\text{Null}(A) = \{\underline{0}\}$, i.e., $A\underline{x} = \underline{0}$ if and only if $\underline{x} = \underline{0}$.
(ii) $\text{rank}\, A = n$
(iii) The linear system $A\underline{x} = \underline{b}$ has no free variables
(iv) The system $A\underline{x} = \underline{b}$ has a unique solution for each $\underline{b} \in \text{Col}(A)$

We can specialize this theorem to square matrices, which allows us to characterize if A is invertible via either its null space or column space:

Theorem If $A \in \mathbb{R}^{n\times n}$, then the following conditions are equivalent:
(i) A is nonsingular
(ii) rank $A = n$
(iii) $\mathrm{Null}(A) = \{0\}$
(iv) $\mathrm{Col}(A) = \mathbb{R}^n$
(v) $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$.

The Superposition Principle

We already saw that for homogeneous systems $A\underline{x} = \underline{0}$, superposition let us generate new solutions by combining known solutions. For inhomogeneous systems $A\underline{x} = \underline{b}$, superposition lets us combine solutions for different RHS.

Suppose we have solutions $\underline{x}_1^*$ and $\underline{x}_2^*$ to $A\underline{x} = \underline{b}_1$ and $A\underline{x} = \underline{b}_2$, respectively. Can I quickly build a solution to $A\underline{x} = c_1\underline{b}_1 + c_2\underline{b}_2$ for some $c_1, c_2 \in \mathbb{R}$?

The answer is yes! We use superposition! Let's try $\underline{x}^* = c_1\underline{x}_1^* + c_2\underline{x}_2^*$:

$$A\underline{x}^* = A(c_1\underline{x}_1^* + c_2\underline{x}_2^*) = c_1(A\underline{x}_1^*) + c_2(A\underline{x}_2^*)$$
$$= c_1\underline{b}_1 + c_2\underline{b}_2 .$$

It worked! This is again the power of linear superposition at play.

Example The system

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

models the mechanical response of a pair of masses connected by springs subject to external forcing.

The solution $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the displacement of the masses and the RHS $\underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ are the applied forces.

For $\underline{f} = \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{x}_1^* = \begin{bmatrix} 4/15 \\ -1/15 \end{bmatrix}$; $\underline{f} = \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{x}_2^* = \begin{bmatrix} -1/15 \\ 4/15 \end{bmatrix}$.

We can write the general solution for $\underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_1\underline{e}_1 + f_2\underline{e}_2$

as $\underline{x}^* = f_1\underline{x}_1^* + f_2\underline{x}_2^*$!

This idea can easily be extended to several RHSs:

If $\underline{x}_1^*, \underline{x}_2^*, \ldots, \underline{x}_k^*$ are solutions to $A\underline{x}=\underline{b}_1, A\underline{x}=\underline{b}_2, \ldots, A\underline{x}=\underline{b}_k$, then for any choice of $c_1, c_2, \ldots, c_k \in \mathbb{R}$, a particular solution to

$$A\underline{x} = c_1\underline{b}_1 + c_2\underline{b}_2 + \cdots + c_k\underline{b}_k \qquad (*)$$

is given by $\underline{x}^* = c_1\underline{x}_1^* + c_2\underline{x}_2^* + \cdots + c_k\underline{x}_k^*$. The general solution to equation $(*)$ is then

$$\underline{x} = \underline{x}^* + \underline{n} = c_1\underline{x}_1^* + c_2\underline{x}_2^* + \cdots + c_k\underline{x}_k^* + \underline{n}$$

where $\underline{n} \in \text{Null}(A)$.

This is exciting! For example, if we know the particular solutions $\underline{x}_1^*, \ldots, \underline{x}_m^*$ to $A\underline{x}=\underline{e}_i$, $i=1,\ldots,m$, where $\underline{e}_1, \ldots, \underline{e}_m$ are the standard basis vectors of $\mathbb{R}^m$, then we can construct a particular solution $\underline{x}^*$ to $A\underline{x}=\underline{b}$ by first writing

$$\underline{b} = b_1\underline{e}_1 + \cdots + b_m\underline{e}_m,$$

to conclude that $\underline{x}^* = b_1\underline{x}_1^* + b_2\underline{x}_2^* + \cdots + b_m\underline{x}_m^*$ is a solution to $A\underline{x}=\underline{b}$.

This is conceptually useful because it tells us how the elements b_i of the RHS $\underline{b}$ affect our solution $\underline{x}^*$.

Practically, it is of limited value however, for example, if A is square, this is just another way of computing A^{-1}. Indeed, the vectors $\underline{x}_1^*, \ldots, \underline{x}_m^*$ are just the columns of A^{-1} (why?), and $\underline{x}^*$ is none other than $\underline{x}^* = A^{-1}\underline{b}$.

Adjoint Systems, Left Null Space, and Row Space

A brief interlude on the matrix transpose:

The transpose A^T of an $m \times n$ matrix A is the $n \times m$ matrix obtained by interchanging its rows and columns. So if $B = A^T$, then $b_{ij} = a_{ji}$.

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Taking the transpose of a column vector gives a row vector:

$$\underline{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \underline{c}^T = [1 \ \ 2 \ \ 3]$$

Here are some properties that are not too hard to check:

- $(A^T)^T = A$ (transpose of transpose brings you back to where you started)
- $(A+B)^T = A^T + B^T$ (transpose and addition commute)
- $(AB)^T = B^T A^T$ (reverse order on products).
- $(A^{-1})^T = (A^T)^{-1} = A^{-T}$ (inverse and transpose commute, provided A^{-1} exists)

A final special case (we will see again next week) is the product of a row vector $\underline{v}^T$ and a column vector $\underline{w}$,

$$\underline{v}^T \underline{w} = (\underline{v}^T \underline{w})^T = \underline{w}^T \underline{v},$$

because the product is a scalar. The transpose of a scalar is itself.

This section explores the properties of the system of linear equations defined by A^T, rather than A. This adjoint system might first appear as some abstract nonsense that only mathematicians care about, but we'll show that it has very practical consequences and interpretations.

Formally, the adjoint to a linear system $A\underline{x} = \underline{b}$ of m equations in n unknowns is the linear system

$$A^T \underline{y} = \underline{f}$$

consisting of n equations in m unknowns $\underline{y} \in \mathbb{R}^m$ and with RHS $\underline{f} \in \mathbb{R}^n$.

Example Consider the linear system $A\underline{x} = \underline{b}$, with coefficient matrix

$$A = \begin{bmatrix} 1 & -3 & 7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{bmatrix},$$

which has transpose $A^T = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ 7 & 5 & -2 \\ 9 & -3 & 6 \end{bmatrix}$.

Thus the adjoint system $A^T\underline{y} = \underline{f}$ is

$$\begin{aligned} y_1 + \quad\quad\; y_3 &= f_1 \\ -3y_1 + y_2 - 2y_3 &= f_2 \\ 7y_1 + 5y_2 - 2y_3 &= f_3 \\ 9y_1 - 3y_2 + 6y_3 &= f_4. \end{aligned}$$

Now at first glance the solutions to $A\underline{x} = \underline{b}$ and the solutions to its adjoint $A^T\underline{y} = \underline{f}$ seem unrelated. We'll soon see some very surprising connections between them that will be revisited in even greater depth later in the course.

We start by introducing the last two fundamental subspaces associated with a matrix A.

The row space (also called coimage) of $A \in \mathbb{R}^{m \times n}$ is the column space of its transpose:

$$\text{Row}(A) = \text{Col}(A^T) = \{ \underline{f} \in \mathbb{R}^n : \underline{f} = A^T\underline{y} \text{ for some } \underline{y} \in \mathbb{R}^m \} \subset \mathbb{R}^n$$

It is called the row space because it is the subspace of $\mathbb{R}^n$ spanned by rows of A (more precisely, by the columns obtained from transposing the rows of A).

The left null space (also called cokernel) of $A \in \mathbb{R}^{m \times n}$ is the null space of its transpose:

$$\text{LNull}(A) = \text{Null}(A^T) = \{ \underline{w} \in \mathbb{R}^m : A^T\underline{w} = 0 \} \subset \mathbb{R}^m$$

It is called the left null space of A because up to taking a transpose, $\text{LNull}(A)$ is composed of row vectors $\underline{w}^T$ satisfying $\underline{w}^T A = \underline{0}^T$.

See online notes for an example where we describe all four subspaces $\text{col}(A)$, $\text{null}(A)$, $\text{Row}(A)$, $\text{LNull}(A)$ for a given system $A\underline{x} = \underline{b}$.

The Fundamental Theorem of Linear Algebra

Theorem: Let A be an $m \times n$ matrix, and let r be its rank. Then

$$\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A) = \mathrm{rank}\, A = \mathrm{rank}\, A^T = r,$$
$$\dim \mathrm{Null}(A) = n - r, \quad \dim \mathrm{LNull}(A) = m - r$$

This theorem says something remarkable. Remember we defined the rank of a matrix A as the number of pivots, which coincides with the number of linearly independent columns of A. Incredibly, the theorem above tells us that this is also equal to the number of linearly independent rows!

This means taking transposes doesn't affect rank: rather rank is intrinsic to a matrix.

Those of you interested in the proof can find it on pp. 114–118 of ALA. We will focus on the implications of this theorem instead.

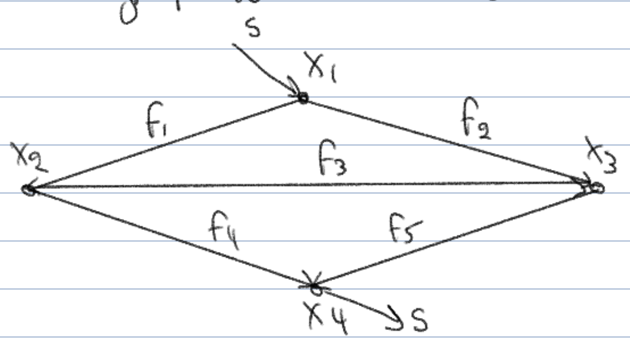
One is the Rank-Nullity theorem, which states that if $A \in \mathbb{R}^{m \times n}$, then

$$\underbrace{\dim \mathrm{Col}(A)}_{\mathrm{rank}(A)} + \underbrace{\dim \mathrm{Null}(A)}_{\mathrm{nullity}(A)} = n$$

The same statement holds true w/ $\mathrm{Col}(A) \to \mathrm{Row}(A)$ and $\mathrm{Null}(A) \to \mathrm{LNull}(A)$.

Network Flows Revisited

Consider the directed graph with 4 nodes and 5 edges



We can associate an incidence matrix A with this graph. Each row corresponds to a node, and each column to an edge, with

$$a_{ij} = \begin{cases} 1 & \text{if edge } j \text{ points to node } i \\ -1 & \text{if edge } j \text{ points from node } i \\ 0 & \text{otherwise.} \end{cases}$$

For our example, $A \in \mathbb{R}^{4\times5}$ and

$$\begin{array}{c|ccccc|c} \text{edges} & 1 & 2 & 3 & 4 & 5 & \text{nodes} \\ \hline \end{array}$$

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

By considering the four fundamental subspaces of A, we can completely understand the properties of our network flow problem.

First, we define the source vector $\underline{s} = \begin{bmatrix} s \\ 0 \\ 0 \\ -s \end{bmatrix}$, which captures external flows entering (+ve entries) the network, and flows leaving the network (-ve entries). These are referred to as sources and sinks, respectively. Here we make sure $\underline{1}^T\underline{s} = 0$.

The flow conservation equations say that flows entering a node must equal flows leaving a node. This can be compactly expressed as

$$A\underline{f} + \underline{s} = 0 \quad \text{or} \quad A\underline{f} = -\underline{s}$$

where $\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \in \mathbb{R}^5$ is the vector of edge flows. If we assume that s_1 and s_4 are given to us (they often are), then the solution set to (*) characterizes all flows compatible with the network and the source vector $\underline{s}$.

First, we see that $A\underline{f} = -\underline{s}$ has a solution if and only if $\underline{s} \in \text{col}(A)$. Let's try to understand when this might be true by computing a basis for col(A).

Notice that col's 1, 2, and 3 are not independent: col 3 = col 2 − col 1.
col's 3, 4, and 5 are not independent: col 5 = col 4 − col 3.
= col 4 − col 2 + col 1

But we have that col's 1, 2, and 4 are independent! And since we can express cols 3 and 5 in terms of them, they span Col(A). We conclude that cols 1, 2 and 4 form a basis for col(A), and dim Col(A) = 3.

But let's look closer: edges 1, 2, and 3 form a loop in the graph, and wouldn't you know it, edges 3, 4, and 5 do too!

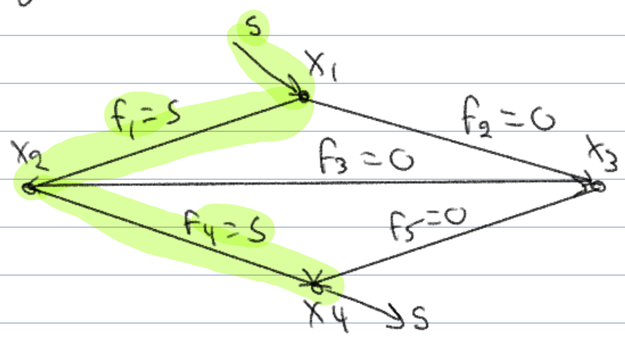
In contrast, edges 1, 2, and 4 form a tree, which has no loops!

This tells us that the edges of any tree in our graph gives us independent columns!

So we now check when $\underline{s} \in \mathrm{Col}(A)$ by checking if

$$\begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 4} \\ \end{array}$$
$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_4 \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ 0 \\ +s \end{bmatrix} \Rightarrow \begin{array}{l} -f_1 - f_2 = -s \\ f_1 - f_4 = 0 \\ f_2 = 0 \\ f_4 = +s \end{array} \Rightarrow \underline{f}^* = \begin{bmatrix} s \\ 0 \\ 0 \\ s \\ 0 \end{bmatrix}$$

which has solution $f_1 = f_4 = s$ and $f_2 = 0$. This corresponds to putting all the flow on edges 1 and 4:



Of course, there are other ways to distribute the flow s to satisfy $A\underline{f} = -\underline{s}$. That's where the null space of A comes in!

Next, let's look at $\mathrm{Null}(A)$. This is the solution set to $A\underline{f} = \underline{0}$, which captures flow conservation in the absence of external sources. This corresponds to (flow in) − (flow out) = 0 at each node: this is called Kirchoff's current law in electric circuits!

We already noticed that col 3 = col 2 − col 1, so one solution to $A\underline{f} = 0$ is $\underline{f}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ (check it!), which corresponds to going around the 1, 2, 3 loop!

Similarly, col 5 = col 4 − col 3, giving $\underline{f}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ (check it), corresponding to the 3, 4, 5 loop! $\underline{f}_1$ and $\underline{f}_2$ are linearly independent, and we know that $\dim \mathrm{Null}(A) = 5 - \dim \mathrm{Col}(A) = 5 - 3 = 2$, so $\underline{f}_1$ and $\underline{f}_2$ form a basis!

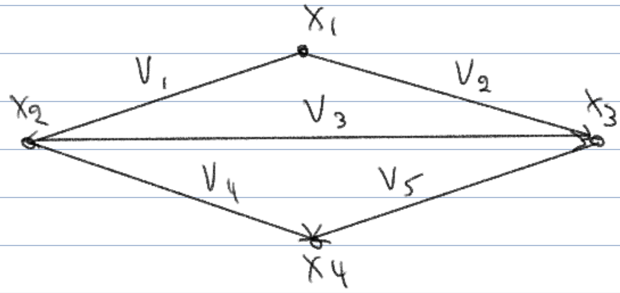
our variables $\underline{f}$ live in $\mathbb{R}^5$

We can therefore write the general solution to $A\underline{f} = -\underline{s}$ as

$$\underline{f} = \underline{f}^* + c_1 \underline{f}_1 + c_2 \underline{f}_2 .$$

Can you guess why elements $\underline{n} \in \mathrm{Null}(A)$ are called circulations?

Now, let us revisit our graph, but instead of flows, let's worry about potential differences, or voltages, across nodes



Solving $A^T\underline{x} = \underline{v}$ tells us what potentials we need to put on the nodes to achieve the desired voltages. For example, the first row of $A^T\underline{x} = \underline{v}$ reads $-x_1 + x_2 = v_1$. This is Kirchoff's voltage law!

Let's start with Null(A^T), which we find by setting $\underline{v} = \underline{0}$. The first equation says $x_1 = x_2$, the second $x_1 = x_3$, the fourth $x_2 = x_4$. We conclude that all four unknowns must take the same value, i.e., $x_1 = x_2 = x_3 = x_4 = c$.

This means Null(A^T) is a line in $\mathbb{R}^4$ spanned by $\underline{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. The rank of A must be $4-1=3$, which we saw was true above!

The row space of A is the column space of A^T. There must be 3 independent columns of A^T, so let's try to find them by inspection. The first three columns of A^T are

$$\underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}}_{\underline{v}_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{\underline{v}_3}$$

which can be checked to be linearly independent quickly. Therefore only voltage configurations $\underline{v}$ lying in Span$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ can be encoded on this graph.

Challenge question: Can you interpret what this statement means physically

$$\underline{v} \in col(A^T) \text{ if and only if } \underline{f}_1^T\underline{v} = 0 \text{ and } \underline{f}_2^T\underline{v} = 0$$

where $\underline{f}_1$ and $\underline{f}_2$ are the basis elements for Null(A)?

Answer: The basis elements $\underline{f}_1$ and $\underline{f}_2$ encode loops in the graph. This says that $\underline{V}$ is a valid voltage profile if and only if summing voltages along a loop equals zero. This is another way of stating Kirchoff's Voltage law.

We will understand where this statement comes from in the next couple of lectures!